Stationary Gravitational and Electromagnetic Fields in General Relativity

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Abstract

Stationary gravitational and electromagnetic fields and their symmetries are considered in the framework of general relativity.

1 INTRODUCTION

Following Ernst[1, 2], and others[3, 4], we generalize Newtonian and electromagnetic potentials in accordance with Einstein's general theory of relativity.

In Sec. 2, we review the formalism.[3] Based on this, in Sec. 3, we define general-relativistic analogs of Newtonian and electromagnetic potentials, ξ and ζ. The field equations for these potentials are invariant under a group of transformations, which is shown to be the only subgroup of the Kinnersley group[4] that conserves the boundary conditions for the potentials. The physical meaning of the group is fully investigated. In the following sections, various cases are examined in which the equations are solvable, and are summarized in Figure. New solutions obtained are those representing a set of arbitrarily spinning Tomimatsu-Sato sources in neutral equilibrium (Sec. 6). Also, we have obtained the correct forms of what were known as the five-parameter solutions (Sec. 7). Possibility of their stationary extensions is touched upon (Sec. 7). Finally, in Sec. 10, possible relevance of gravitation to classical electron theory is suggested. Throughout we adopt the tensor convention of Ref. 5.

2 STATIONARY GRAVITATIONAL AND ELECTROMAGNETIC FIELDS[3]

Throughout we are concerned with spacetimes that allow at least one Killing vector. Put in different words, the spacetime is stationary, i.e., metric coefficients $g_{\mu\nu}$ are independent of the time, e. g. We then want to solve the Einstein equation (in geometrized units:[5] $G = c = 1$)

$$R_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2. 1}$$

where the only source of gravity is the electromagnetic stress-energy tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\right). \tag{2. 2}$$

The Maxwell tensor $F_{\mu\nu}$ satisfies the Maxwell equations in the curved spacetime

$$F^{\mu\nu}{}_{;\nu} = 0, \tag{2. 3}$$

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0. \tag{2. 4}$$

Eq. (2. 1), in turn, tells the spacetime how to curve. We are thus solving the coupled Einstein-Maxwell equations. Mass and charge, supposed to be distributed in a measure-zero subset of space, manifest themselves as "wormholes" or singularities of the field equations.

We write the metric in the form[3]

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$
$$= -f(dx^0 + w_m dx^m)^2 + f^{-1} h_{mn} dx^m dx^n, \qquad (2.\ 5)$$

where Greek indices run from 0 to 3, and Latin, from 1 to 3. Our convention is such that in the Minkowski spacetime $f = 1$, $w_m = 0$, and $h_{mn} = \delta_{mn}$. All the metric functions, f, w_m, and h_{mn} are supposed to depend only on x^1, x^2, and x^3.

We write the inverse of the matrix h_{mn} as $h^{\hat{m}\hat{n}}$. Upper indices raised by $h^{\hat{m}\hat{n}}$ have hats on, e. g.,

$$w^{\hat{m}} = h^{\hat{m}\hat{n}} w_m, \qquad (2.\ 6)$$

whereas upper indices raised by $g^{\mu\nu}$, the inverse of $g_{\mu\nu}$, are bareheaded as usual. The square of a 3-vector is defined, e.g., by

$$w^2 = w^{\hat{m}} w_m. \qquad (2.\ 7)$$

Simple calculation shows that

$$g^{00} = fw^2 - f^{-1}, \qquad (2.\ 8)$$
$$g^{0m} = -\ fw^{\hat{m}}, \qquad (2.\ 9)$$
$$g^{mn} = fh^{\hat{m}\hat{n}}, \qquad (2.\ 10)$$
$$g = \det \| g_{\mu\nu} \| = -\ f^{-2} h, \qquad (2.\ 11)$$

where

$$h = \det \| h_{mn} \|. \qquad (2.\ 12)$$

We define the electromagnetic vector potential A_μ and the magnetic potential Φ by

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \qquad (2.13)$$

$$F^{mn} = -f\varepsilon^{\hat{m}\hat{n}\hat{\ell}}\Phi_{,\ell}, \qquad (2.14)$$

where the 3-space Levi-Civita tensors are

$$\varepsilon_{mn\ell} = h^{1/2}[mn\ell], \qquad (2.15)$$

$$\varepsilon^{\hat{m}\hat{n}\hat{\ell}} = h^{-1/2}[mn\ell]. \qquad (2.16)$$

The fully antisymmetric symbol $[mnl]$ is defined by $[123] = 1$, etc. All the components of the Maxwell tensor are obtained by A_o and Φ,

$$F_{no} = A_{o,n}, \qquad (2.17)$$

$$F^{on} = fw_m\varepsilon^{\hat{m}\hat{n}\hat{\ell}}\Phi_{,\ell} + A_{o,m}h^{\hat{m}\hat{n}}, \qquad (2.18)$$

$$F_{mn} = -f^{-1}\varepsilon_{mn\ell}\Phi_{,}{}^{\hat{\ell}} + A_{o,m}w_n - A_{o,n}w_m. \qquad (2.19)$$

We define the electromagnetic complex potential Ψ by[3]

$$\Psi = A_o + i\Phi. \qquad (2.20)$$

To find an appropriate definition for the corresponding gravitational complex potential, we first note that w_m is arbitrary up to an additive gradient. However, the following "torsion vector"[3] is invariant,

$$\tau^{\hat{m}} = f^2 \varepsilon^{\hat{m}\hat{n}\hat{\ell}} w_{n,\ell}. \tag{2. 21}$$

The factor f^2 has been put in for later convenience. We then want to put the Einstein equation, Eq. (2. 1), in terms of Ψ, f, $\tau^{\hat{m}}$, and h_{mn}. First, we calculate the Ricci tensor,

$$R_{\mu\nu} = (-g)^{-1/2} [(-g)^{1/2} \Gamma^{\alpha}{}_{\mu\nu}]_{,\alpha}$$

$$- [\ln(-g)^{1/2}]_{,\mu\nu} - \Gamma^{\alpha}{}_{\beta\nu} \Gamma^{\beta}{}_{\mu\alpha}, \tag{2. 22}$$

with

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}). \tag{2. 23}$$

The task is straightforward but tedious. The results are

$$R_{00} = \frac{1}{2} f \nabla^2 f + \frac{1}{2}\tau^2 - \frac{1}{2}(\nabla f)^2, \tag{2. 24}$$

$$R_0{}^n = -\frac{1}{2} f \varepsilon^{\hat{n}\hat{a}\hat{b}} \tau_{a,b}, \tag{2. 25}$$

$$R^{mn} = \hat{R}^{mn} + h^{\hat{m}\hat{n}} R_{00} - \frac{1}{2}\tau^{\hat{m}}\tau^{\hat{n}} - \frac{1}{2} f_,{}^{\hat{m}} f_,{}^{\hat{n}}, \tag{2. 26}$$

where $\hat{R}_{mn}$ is the Ricci tensor calculated from the 3-space metric,

$$d\ell^2 = h_{mn}\, dx^m dx^n. \tag{2. 27}$$

Gradients are also taken in this 3-space. Similarly, we express the stress-energy tensor in terms of Φ and A_0,

$$8\pi f^{-1} T_{00} = (\nabla\Phi)^2 + (\nabla A_0)^2, \qquad (2.\ 28)$$

$$4\pi T_0{}^m = -f A_{0,a}\, \varepsilon^{\hat{a}\hat{m}\hat{s}}\, \Phi_{,s}\,, \qquad (2.\ 29)$$

$$-4\pi f^{-1} T^{mn} = \Phi_,{}^{\hat{m}}\, \Phi_,{}^{\hat{n}} + A_{0,}{}^{\hat{m}} A_{0,}{}^{\hat{n}} - \frac{1}{2} h^{\hat{m}\hat{n}} [(\nabla\Phi)^2 + (\nabla A_0)^2]. \qquad (2.\ 30)$$

Then $R_0{}^m = 8\pi T_0{}^m$ gives

$$\nabla \times [\vec{\tau} + i(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*)] = 0\,, \qquad (2.\ 31)$$

so we can define a real "twist potential" ψ by[3, 4]

$$\nabla \psi = \vec{\tau} + i(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*). \qquad (2.\ 32)$$

The gravitational complex potential is defined, at last, by[3]

$$\mathcal{E} = f - \Psi\Psi^* + i\psi. \qquad (2.\ 33)$$

Then $R_{00} = 8\pi T_{00}$ gives[3]

$$(\mathrm{Re}\,\mathcal{E} + \Psi\Psi^*)\nabla^2 \mathcal{E} = \nabla\mathcal{E} \cdot (\nabla\mathcal{E} + 2\Psi^* \nabla \Psi)\,, \qquad (2.\ 34)$$

while the remaining part of the Einstein equation takes on the form,[3]

$$(\mathrm{Re}\,\mathcal{E} + \Psi\Psi^*)^2 \hat{R}_{mn} = \frac{1}{2} \mathcal{E}_{,(m} \mathcal{E}^*_{,n)} + \Psi \mathcal{E}_{,(m} \Psi^*_{,n)}$$

$$+ \Psi^* \mathcal{E}^*_{,(m} \Psi_{,n)} - (\mathcal{E} + \mathcal{E}^*) \Psi_{,(m} \Psi^*_{,n)}. \qquad (2.\ 35)$$

On the other hand, the Maxwell equation, Eq. (2. 3), can be written as[3]

$$(\mathrm{Re}\,\mathcal{E}+\Psi\Psi^{*})\nabla^{2}\Psi=\nabla\Psi\cdot(\nabla\mathcal{E}+2\Psi^{*}\nabla\Psi). \qquad (2.\ 36)$$

The outcome of this section is summarized as follows. Stationary solutions of the Einstein-Maxwell equations are obtained by solving Eqs. (2. 34) and (2. 36) in a curved 3-space. Eq. (2. 35), in turn, tells the 3-space how to curve.

3 FUNDAMENTAL SYMMETRY GROUP

In the preceding section we followed Israel and Wilson[3] and transformed the entire Einstein-Maxwell equations into 3-space differential equqtions involving two complex potentials, $\mathcal{E}$ and Ψ. However, these potentials are not very useful in actual calculation, because (1) in the field equations the appearance of $\mathcal{E}$ and Ψ is not very symmetric, and (2) if we set $\psi \to 0$ and $\Psi \to 0$ at spatial infinity, then $\mathcal{E} \to 1$, so the boundary conditions for $\mathcal{E}$ and Ψ are not symmetric, either.

For our purpose it is convenient to define two alternative complex potentials ξ and ζ by

$$\mathcal{E} = \frac{1-\xi}{1+\xi}, \tag{3. 1}$$

$$\Psi = \frac{\zeta}{1+\xi}. \tag{3. 2}$$

(For other definitions, see Ernst[1, 2] and Kinnersley[4].) Then the field equations take on the forms,

$$(\xi\xi^*-\zeta\zeta^*-1)\nabla^2\xi = 2(\xi^*\nabla\xi-\zeta^*\nabla\zeta)\cdot\nabla\xi, \tag{3. 3}$$

$$(\xi\xi^*-\zeta\zeta^*-1)\nabla^2\zeta = 2(\xi^*\nabla\xi-\zeta^*\nabla\zeta)\cdot\nabla\zeta, \tag{3. 4}$$

and

$$(\xi\xi^*-\zeta\zeta^*-1)^2\hat{R}_{mn} = [(\xi_{,m}\zeta-\xi\zeta_{,m})^*(\xi_{,n}\zeta-\xi\zeta_{,n})$$

$$+\xi^*_{,m}\xi_{,n}-\zeta^*_{,m}\zeta_{,n}] + [\ m \rightleftarrows n\ ]. \tag{3. 5}$$

These equations are invariant under

(I) $$\xi \to \xi \cosh\chi + \zeta \sinh\chi\,, \tag{3. 6a}$$

$$\zeta \to \xi \sinh\chi + \zeta\cosh\chi\,; \tag{3. 6b}$$

(II) $$\xi \to e^{i\alpha}\xi\,, \quad \zeta \to \zeta\,; \tag{3. 7}$$

(III) $$\xi \to \xi\,, \quad \zeta \to e^{i\beta}\zeta\,; \tag{3. 8}$$

and any combination thereof, where χ, α, and β are real constants. We shall say that these three transformations form the fundamental symmetry group of stationary spacetimes. It can be shown that this is the only subgroup that conserves the correct asymptotic conditions, $\mathcal{E} \to 1$ and $\Psi \to 0$.

To see the physical meaning of the potentials and the symmetry group, we first note that, to order r^{-1},

$$\zeta \approx \Psi \approx \frac{-Q_e + iQ_m}{r}\,, \tag{3. 9}$$

far from the origin of coordinates, where Q_e and Q_m are the total electric and magnetic charges, respectively, of the sources, i. e., singularities of the field equations. Similarly, from

$$\begin{aligned} f &= \frac{1+\zeta\zeta^{*}-\xi\xi^{*}}{|1+\xi|^2} \\ &= 1-\frac{2\,\mathrm{Re}\,\xi + 2\xi\xi^{*} - \zeta\zeta}{|1+\xi|^2}\,, \end{aligned} \tag{3. 10}$$

and Newtonial approximation,

$$f \approx 1 - \frac{2M}{r}\,, \tag{3. 11}$$

where M is mass, we see that Re ξ corresponds to Newtonian potential. We thus write

$$\xi \approx \frac{M + iN}{r}, \tag{3. 12}$$

and see what N means, by noting

$$\begin{aligned} & i(\xi\xi^* - \zeta\zeta^* - 1)^2 \nabla \times \vec{w} \\ & = [(1+\xi^*)(1+\xi^* + \zeta\zeta^*)\nabla\xi - |1+\xi|^2 \zeta^* \nabla \zeta] \\ & \quad - [\text{complex conjugation}]. \end{aligned} \tag{3. 13}$$

Asymptotically,

$$\nabla \times \vec{w} \approx 2\,\mathrm{Im}\,\nabla\xi \approx -2N\frac{\vec{r}}{r^3}, \tag{3. 14}$$

which is analogous to the old problem of solving for the magnetic field vector induced by a spherically symmetric outgoing current. There is no globally nonsingular solution; current must be supplied to the origin via a line singularity. One solution is

$$w_m dx^m \approx 2N \cos\theta \, d\phi, \tag{3. 15}$$

which is singular on the z-axis. Thus, although Eq. (3. 12) with nonvanishing N is a perfectly good solution to the field equations, it is excluded since no asymptotically Minkowski metric can be introduced. Furthermore it is shown that the presence of N violates causality by allowing closed timelike loops. Historically, a generalization of the Schwarzschild metric with such properties was first obtained by Newman et al. by a different route[6].

It is now clear that transformation (I) mixes electlic charge with mass and magnetic charge with N, (II), gravitational duality rotation, blends mass with N, and (III), electromagnetic duality rotation, mixes electricity with magnetism. Note that these transformations do not alter 3-geometry.

For further discussion, see Sec. 10.

4 CONSTRAINTS NOS. 1 - 3

We have seen that the Einstein-Maxwell equations are conveniently put in terms of gravitational and electromagnetic potentials ξ and ζ. To order r^{-1}, the potentials are, clearly, proportional to each other. What if we fix all the higher moments similarly?

Constraint No. 1:

$$\zeta = c\xi, \tag{4. 1}$$

where

$$c = -Q/M \tag{4. 2}$$

is a complex constant. This fixes all the higher moments and, if c is real, the gyromagnetic ratio is that of the Dirac electron (see Sec. 5). If $|c|<1$, then define

$$\xi_0 = (1-cc^*)^{1/2}\xi. \tag{4. 3}$$

Then the field equations, Eqs. (3. 3) through (3. 5), reduce to[1, 2]

$$(\xi_0\xi_0^*-1)\nabla^2\xi_0 = 2\xi_0^*\nabla\xi_0\cdot\nabla\xi_0, \tag{4. 4}$$

and

$$(\xi_0\xi_0^*-1)^2\hat{R}_{mn} = 2\,\mathrm{Re}\,\xi_{0,m}\xi_{0,n}^*. \tag{4. 5}$$

If $|c|>1$, then define $\xi_0 = (cc^*-1)^{1/2}\xi$, and both the field equations and formulas for metric functions remain unchanged excepting sign changes of terms quadratic in ξ_0, thus leaving the resulting metric unaltered. The case $|c| = 1$ is obtained by taking the limit, or else see below.

We note in passing that Eq. (4. 4), the Ernst equation[1, 2], and Eq. (4. 5) are invariant under (1) $\xi_0 \to \xi_0^*$, (2) $\xi_0 \to \xi_0^{-1}$, (3) $\xi_0 \to e^{i\alpha}\xi_0$, (4)

$$\xi_0 \to \frac{\sinh\beta + \xi_0\cosh\beta}{\cosh\beta + \xi_0\sinh\beta}, \tag{4. 6}$$

and combinations thereof, where α and β are real constants. Just as in quantum mechanics, $\xi_0 \to \xi_0^*$ reverses spin sense. $\xi_0 \to e^{i\alpha}\xi_0$ is gravitational duality rotation (see Sec. 3). Others are useful only for computational purposes.

Looking at Eqs. (4. 3) and (4. 5), we see that when $|c| = 1$, the 3-space Ricci tensor vanishes, so

Constraint No. 2:

$$\zeta = e^{i\alpha}\xi, \tag{4. 7}$$

which implies that $-Q = e^{i\alpha}M$, where α is a real constant. The field equation becomes[3, 7]

$$\nabla^2\xi = 0. \tag{4. 8}$$

Since $\hat{R}_{mn}$ vanishes, Eq. (4. 8) is a flat-space Laplace equation, so superposition is possible. This means that gravitational pull exactly balances electromagnetic push.

We may use variables other than ξ_0; e. g.,[1]

$$\xi_0 = e^{i\nu} \tag{4. 9}$$

leads to

$$i\nabla^2\nu = \coth(\mathrm{Im}\nu)\,\nabla\nu\cdot\nabla\nu, \qquad (4.\ 10)$$

and[1, 8]

$$\xi_0 = -\tanh\psi = \frac{1-e^{2\psi}}{1+e^{2\psi}} \qquad (4.\ 11)$$

results in[8]

$$\nabla^2\psi = 2i\tan(2\,\mathrm{Im}\psi)\,\nabla\psi\cdot\nabla\psi\,. \qquad (4.\ 12)$$

The right-hand side of Eq. (4. 12) vanishes if ψ and hence ξ_0 are real, so

<u>Constraint No. 3</u> is Constraint No. 1 with ξ_0 real.[1] Then we define a real potential ψ by Eq. (4. 11) or

$$\psi = \frac{1}{2}\ln\frac{1-\xi_0}{1+\xi_0}, \qquad (4.\ 13)$$

which looks like Eq. (4. 8) but is a <u>curved</u>-space Laplace equation.

We shall introduce a still stronger constraint in the next section.

5 CONSTRAINT NO. 4: AXIAL SYMMETRY

Constraint No. 4 demands that the metric be axisymmetric, and hence written as[9, 10, 11]

$$ds^2 = -f(dt+\omega d\phi)^2 + f^{-1}[e^{2\gamma}(dz^2+d\rho^2)+\rho^2 d\phi^2] \qquad (5.\ 1)$$

by suitably choosing the coordinates, where f, ω, and γ depend only on z and ρ. Simple calculation then shows that, for any function of z and ρ,

$$\nabla^2 A = e^{-2\gamma}(A_{,zz} + A_{,\rho\rho} + \rho^{-1}A_{,\rho}), \qquad (5.\ 2)$$

$$\nabla A \cdot \nabla B = e^{-2\gamma}(A_{,z}B_{,z} + A_{,\rho}B_{,\rho}), \qquad (5.\ 3)$$

and so all the 3-space differential equations we encounter in this work become those of flat space. (This does not, of course, mean that space is flat.)

The only nonvanishing 3-space Ricci components are

$$\hat{R}_{zz} = -\rho^{-1}\gamma_{,\rho} - \gamma_{,zz} - \gamma_{,\rho\rho}, \qquad (5.\ 4)$$

$$\hat{R}_{\rho\rho} = \rho^{-1}\gamma_{,\rho} - \gamma_{,zz} - \gamma_{,\rho\rho}, \qquad (5.\ 5)$$

$$\hat{R}_{z\rho} = \hat{R}_{\rho z} = \rho^{-1}\gamma_{,z}. \qquad (5.\ 6)$$

From these we obtain

$$\gamma_{,z} = \rho\hat{R}_{z\rho}, \qquad (5.\ 7)$$

$$\gamma_{,\rho} = \frac{1}{2}\rho(\hat{R}_{\rho\rho} - \hat{R}_{zz}), \qquad (5.\ 8)$$

from which and Eq. (3. 5) γ is calculated. In particular, when Constraint No. 1 is present,

$$\gamma_{,z} = \frac{2\rho}{(\xi_0\xi_0^*-1)^2} \,\mathrm{Re}\,(\xi_{0,z}\xi_{0,\rho}^*), \tag{5. 9}$$

$$\gamma_{,\rho} = \frac{\rho}{(\xi_0\xi_0^*-1)^2} (\xi_{0,\rho}\xi_{0,\rho}^* - \xi_{0,z}\xi_{0,z}^*). \tag{5. 10}$$

On the other hand, ω is given in general by

$$\omega_{,z} = f^{-2}\rho\,\mathrm{Im}(\mathcal{E}_{,\rho} + 2\Phi^*\Phi_{,\rho}), \tag{5. 11}$$

$$\omega_{,\rho} = -f^{-2}\rho\,\mathrm{Im}(\mathcal{E}_{,z} + 2\Phi^*\Phi_{,z}), \tag{5. 12}$$

and when Constraint No. 1 is present, by

$$\omega_{,z} = \frac{-2\rho}{(\xi_0\xi_0^*-1)^2}\mathrm{Im}[(1+\xi^*)(1+(1+cc^*)\xi^*)\xi_{,\rho}], \tag{5. 13}$$

$$\omega_{,\rho} = \frac{2\rho}{(\xi_0\xi_0^*-1)^2}\mathrm{Im}[(1+\xi^*)(1+(1+cc^*)\xi^*)\xi_{,z}]. \tag{5. 14}$$

When approximate spherical symmetry is present, spheroidal coordinates[12] are much more useful. <u>Prolate</u> spheroidal coordinates (x, y) are defined by

$$\rho = \kappa(x^2-1)^{1/2}(1-y^2)^{1/2}, \tag{5. 15}$$

$$z = \kappa x y, \tag{5. 16}$$

where κ is a constant. We note that

$$\kappa^2(x^2-y^2)\nabla^2 A = [(x^2-1)A_{,x}]_{,x} + [(1-y^2)A_{,y}]_{,y}, \tag{5. 17}$$

$$\kappa^2(x^2-y^2)\nabla A\cdot\nabla B=(x^2-1)A_{,x}B_{,x}+(1-y^2)A_{,y}B_{,y}\,, \qquad (5.\ 18)$$

and hence if $\xi_0(x, y)$ is a solution of Eq. (4. 4), e. g., then so is $\xi_0(y, x)$. Similarly, oblate spheroidal coordinates $(\tilde{x}, y)$ are defined by

$$\rho=\tilde{\kappa}(\tilde{x}^2+1)^{1/2}(1-y^2)^{1/2}, \qquad (5.\ 19)$$

$$z=\tilde{\kappa}\tilde{x}y\,, \qquad (5.\ 20)$$

and are automatically effected from prolate ones by letting

$$\tilde{x}=ix\,, \qquad (5.\ 21)$$

$$\tilde{\kappa}=-i\kappa\,. \qquad (5.\ 22)$$

With prolate coordinates the metric assumes the form

$$ds^2=-f(dt+\omega d\phi)^2+f^{-1}[e^{2\gamma}\kappa^2(x^2-y^2)\cdot$$

$$\cdot\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+\kappa^2(x^2-1)(1-y^2)d\phi^2\,]. \qquad (5.\ 23)$$

Metric functions are calculated, in the case of Constraint No. 1 in particular, by

$$\gamma_{,x}=\frac{x(1-y^2)}{(\xi_0\xi_0^*-1)^2(x^2-y^2)}\left[(x^2-1)\xi_{0,x}\xi_{0,x}^*-(1-y^2)\xi_{0,y}\xi_{0,y}^*\right.$$

$$\left.-2\frac{y}{x}(x^2-1)\,\mathrm{Re}\,(\xi_{0,x}\xi_{0,y}^*)\,\right], \qquad (5.\ 24)$$

$$\gamma_{,y} = \frac{(x^2-1)y}{(\xi_0\xi_0^*-1)^2(x^2-y^2)}\left[(x^2-1)\xi_{0,x}\xi^*_{0,x} - (1-y^2)\xi_{0,y}\xi^*_{0,y}\right.$$

$$\left. + 2\frac{x}{y}(1-y^2)\,\mathrm{Re}\,(\xi_{0,x}\xi^*_{0,y})\right], \tag{5. 25}$$

$$\omega_{,x} = \frac{2\kappa(1-y^2)}{(\xi_0\xi_0^*-1)^2}\,\mathrm{Im}\,[(1+\xi^*)(1+(1+cc^*)\xi^*)\xi_{,y}], \tag{5. 26}$$

$$\omega_{,y} = \frac{-2\kappa(x^2-1)}{(\xi_0\xi_0^*-1)^2}\,\mathrm{Im}\,[(1+\xi^*)(1+(1+cc^*)\xi^*)\xi_{,x}]. \tag{5. 27}$$

The electromagnetic 2-form is given by

$$\begin{aligned} F &\equiv E_x\, dx\wedge dt + \cdots + B_x\, dy\wedge dz + \cdots \\ &= A_{0,z}\, dz\wedge dt + A_{0,\rho}\, d\rho\wedge dt + A_{0,\phi}\, d\phi\wedge dt \\ &\quad - (f^{-1}\rho\Phi_{,z} - A_{0,\rho}\omega)\, d\rho\wedge d\phi \\ &\quad - (f^{-1}\rho\Phi_{,\rho} + A_{0,z}\omega)\, d\phi\wedge dz \\ &\quad - f^{-1}\rho^{-1}e^{2\gamma}\Phi_{,\phi}\, dz\wedge d\rho . \end{aligned} \tag{5. 28}$$

The asymptotic behavior of ω is known[5] to be

$$\omega \approx 2J\sin^2\theta/r . \tag{5. 29}$$

From this and Eqs. (5. 26) and (5. 27) (identify $y = \cos\theta$ and $\kappa x \approx r$), we find that

$$\mathrm{Im}\,\xi \approx -J\cos\theta/r^2, \tag{5. 30}$$

where J is the source's angular momentum. If N is present in

Eq. (3. 12), then

$$\omega\, dt\, d\phi \approx 2N\cos\theta\, dt\, d\phi = \frac{2N\cos\theta}{\rho} dt \cdot \rho\, d\phi , \tag{5. 31}$$

which is singular on the z-axis.

If there is Constraint No. 1, then

$$\Phi \approx \operatorname{Im} \zeta = c \operatorname{Im} \xi \approx \frac{Qa\cos\theta}{r^2} , \tag{5. 32}$$

where c, assumed to be real here, is given by Eq. (4. 2), and

$$a = J/M. \tag{5. 33}$$

The electromagnetic fields in the usual spherical orthonormal frame are

$$E_{\hat{r}} \approx Q/r^2 , \tag{5. 34}$$

$$B_{\hat{r}} \approx 2Qa \cos\theta / r^3 , \tag{5. 35}$$

$$B_{\hat{\theta}} \approx Qa \sin\theta / r^3 , \tag{5. 36}$$

These reveal that Q is indeed the charge, and that

$$\mathfrak{M} = Qa \tag{5. 37}$$

is the magnetic dipole moment. Notice that the gyromagnetic ratio

$$\mathfrak{M} / J = Q/M \tag{5. 38}$$

is equal to that of the Dirac electron! This fact has been known for the Kerr-Newman geometry, but is actually much more general. (See Sec. 10.)

6 SOME KNOWN SOLUTIONS

Constraints Nos. 3 and 4 together result in a flat-space Laplace equation, Eq. (4. 14). In this case it is easily shown, and is given here for later use, that if ξ_0 and ξ_0' are solutions, then so is

$$\xi_0'' = \frac{\xi_0 + \xi_0'}{1 + \xi_0 \xi_0'}. \tag{6. 1}$$

The general solution can be written as[1]

$$\psi = \sum_\ell \alpha_\ell Q_\ell(x) P_\ell(y), \tag{6. 2}$$

a superposition of Legendre functions. In the case of pure $\ell = 0$, an interesting class of the Weyl solutions[12,13,14]

$$\psi = \frac{\delta}{2} \ln \frac{x-1}{x+1}, \tag{6. 3}$$

or

$$\xi_0 = \frac{(x+1)^\delta - (x-1)^\delta}{(x+1)^\delta + (x-1)^\delta}, \tag{6. 4}$$

results. When $\delta = 1$,

$$\xi_0 = x^{-1}, \tag{6. 5}$$

which can be shown[1] to be exactly equivalent to the Schwarzschild (or its charged counterpart, Reissner-Nordstrøm) metric. Rather than constructing its metric form here, however, we go on to the more general Kerr-Newman metric, of which the Schwarzschild metric is a special case.

$$\xi_0^{-1} = px + iqy \tag{6. 6}$$

fulfills the required asymptotic behavior, and is indeed a solution to Eq. (4. 4) if[1]

$$p^2 + q^2 = 1. \tag{6. 7}$$

It is a straightforward work to find the metric form[2] using formulas given in Sec. 5. After identifying

$$x = (r - M)(M^2 - Q^2)^{-1/2}, \tag{6. 8}$$

$$y = \cos\theta, \tag{6. 9}$$

$$\kappa = (M^2 - Q^2)^{1/2}, \tag{6. 10}$$

$$q = (M^2 - Q^2)^{-1/2} a, \tag{6. 11}$$

where a is specific angular momentum, Eq. (5. 33), we obtain the Kerr-Newman metric[15,16] in the Boyer-Lindquist[17] coordinates:

$$ds^2 = -\frac{\Delta}{R^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{R^2}[(r^2+a^2)d\phi - a dt]^2$$

$$+ \frac{R^2}{\Delta}dr^2 + R^2 d\theta^2, \tag{6. 12}$$

with

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \tag{6. 13}$$

$$R^2 = r^2 + a^2\cos^2\theta. \tag{6. 14}$$

When q = 0, Eq. (6. 12) reduces to the Schwarzschild metric. Likewise, we want to find solutions of the form (see Eqs. (6. 4) and (5. 30)),

$$\xi_0 = \frac{(x+1)^\delta - (x-1)^\delta}{(x+1)^\delta + (x-1)^\delta} - iJ\frac{y}{x^2} + \cdots, \tag{6. 15}$$

for other values of δ. ($\delta = 1$ is the Kerr-Newman metric.) Such solutions are collectively named after Tomimatsu and Sato[18,14] (TS), although only the $\delta = 1$, 2, 3, and 4 cases are explicitly known. (However, see Refs. 19,20). The $\delta = 2$ TS metric, e. g., is calculated from[18,14]

$$\xi_0 = \frac{2px(x^2-1) + 2iqy(1-y^2)}{p^2x^4 + q^2y^4 - 1 + 2ipqxy(x^2-y^2)}. \tag{6. 16}$$

Detailed study[14] shows that it has a nonsingular event horizon and a ring singularity outside the horizon.

We note that in either the Kerr or the TS case, despite the look of Eq. (6. 7), we can let q take on any value by analytic continuation. In other words, we go over to oblate coordinates, Eqs. (5. 19)ff., by transforming

$$x \to ix, \qquad p \to -ip. \tag{6. 17}$$

Then Eq. (6. 7) becomes

$$-p^2 + q^2 = 1. \tag{6. 18}$$

Now, if we infinitely increase both p and q in Eq. (4. 4), with $-p^2 + q^2$ fixed, then the right-hand side decrease faster, and in the limit we arrive at Eq. (4. 8). This procedure gives the solution of Eq. (4. 8) that corresponds, e. g., to the $Q^2 = M^2$ limit of Eq. (6. 6),[3,7]

$$\xi = \frac{M}{a(x+iy)}, \tag{6. 19}$$

in oblate spheroidal coordinates,

$$\rho = a(x^2+1)^{1/2}(1-y^2)^{1/2}, \tag{6. 20}$$

$$z = axy, \tag{6. 21}$$

where we have set $\kappa = a$. In terms of ρ and z, Eq. (6. 19) takes on a more familiar-looking form,

$$\xi = M/R, \tag{6. 22}$$

with

$$R^2 = \rho^2 + (z + ia)^2, \tag{6. 23}$$

a spherically symmetric Laplace solution with an imaginary origin of coordinates. Identifying

$$x = \frac{r-M}{a}, \quad y = \cos\theta, \tag{6. 24}$$

we recover the Kerr-Newman metric, Eq. (6. 12), with $M^2 = Q^2$. Further, we may construct solutions without axial symmetry by superposing Eq. (6. 22),[3,7]

$$\xi = \Sigma \frac{M_k}{R_k}, \tag{6. 25}$$

where

$$R_k{}^2 = (\vec{r} - \vec{c}_k)^2, \tag{6. 26}$$

$\vec{r}$ is the Euclidean position vector, and $\vec{c}_k$ an arbitrary set of constant, complex vectors. The resulting metric will represent a set of arbitrarily spinning Kerr-Newman sources in neutral equilibrium[3, 7]. Likewise, from Eq. (6. 16) we obtain the $Q^2 = M^2$ $\delta = 2$ TS solution,

$$\xi = \frac{M}{a(x+iy)} + \frac{M(1-ixy)}{a(x+iy)^3}. \tag{6. 27}$$

The first term is just Eq. (6. 19). The second term can be rewritten, with Eqs. (6. 20) and (6. 21), as

$$-i\,\frac{Ma(z+ia)}{R^3}. \tag{6. 28}$$

If we write

$$z + ia = R\cos\Theta, \tag{6. 29}$$

then Eq. (6. 27) becomes

$$\xi = \frac{M}{R} - i\,\frac{Ma\cos\Theta}{R^2}. \tag{6. 30}$$

The same technique may be used for other TS solutions. The results are multipole expansions with an imaginary origin of coordinates. (A similar limit is studied by Kinnersley[21].) Thus, in this limit the TS solutions take on quite transparent forms. Also, just as we did in Eq. (6. 25), we may construct a superposition of Eqs. (6. 22), (6. 30), and other multipole expansions. The resulting metric will represent a set of arbitrarily spinning Kerr-Newman and TS sources in neutral equilibrium.

We must admit that we have no idea as to why each coefficient of such expansions should take on some definite value. Perhaps someday solutions corresponding to any multipole expansion may be found.

7 CONSTRAINT NO. 5

Constraint No. 5 shall be such that both of the potentials ξ and ζ are real.[22] We may define another complex potential by $\xi + i\zeta$. However, to make the resulting equations "manifestly covariant" under the fundamental transformation, Eq. (3. 6), we instead define two real potentials

$$W = \xi + \zeta, \qquad \bar{W} = \xi - \zeta. \tag{7. 1}$$

Instead of writing as above, we use the shorthand notation

$$W = \xi + I\zeta, \tag{7. 2}$$

and understand that

$$I = 1, \qquad \bar{I} = -1. \tag{7. 3}$$

Clearly,

$$I^2 = 1. \tag{7. 4}$$

With this notation, transformation Eq. (3. 6) is written as

$$W \to e^{I\chi}W, \tag{7. 5}$$

where

$$e^{I\chi} = \cosh\chi + I\sinh\chi \tag{7. 6}$$

as is understood by expanding the left-hand side in power series and then using Eq. (7. 4). The field equations, Eqs. (3. 3) through (3. 5) take on the forms

$$(\overline{W}W - 1)\nabla^2 W = (\overline{W}\nabla W + W\nabla\overline{W})\cdot\nabla W, \tag{7. 7}$$

$$(\overline{W}W - 1)^2\hat{R}_{mn} = 2\overline{W}_{,(m}W_{,n)}. \tag{7. 8}$$

In the case of axial symmetry, Eq. (7. 7) is a flat-space equation (see Sec. 5). We may use another potential defined by

$$W = \frac{2X}{\overline{X}X+1}. \tag{7. 9}$$

Then Eqs. (7. 7) and (7. 8) acquire the forms

$$(XX - 1)\nabla^2 X = 2\overline{X}\nabla X\cdot\nabla X, \tag{7. 10}$$

and

$$(XX - 1)^2\hat{R}_{mn} = 8\overline{X}_{,(m}X_{,n)}. \tag{7. 11}$$

We note that when $\zeta = 0$, Eq. (7. 9) is just the addition rule, Eq. (6. 1). Eq. (7. 10) is exactly analogous to Eq. (4. 4), so we can guess solutions to Eq. (7. 10) from known solutions of Eq. (4. 4). E. g., we start with the Kerr solution, Eq. (6. 6) with Eq. (6. 7), and formaly transform $q \to -Iq$, resulting in

$$X^{-1} = px + Iqy, \quad p^2 - q^2 = 1, \tag{7, 12}$$

which is indeed a solution to Eq. (7. 10). From this and Eqs. (7. 9) and (7. 2), we obtain[23]

$$\xi = \frac{2px}{p^2x^2 - q^2y^2 + 1}, \tag{7. 13}$$

$$\zeta = \frac{2qy}{p^2x^2 - q^2y^2 + 1}. \tag{7. 14}$$

When q = 0, these reduce to the $\delta = 2$ member of Eq. (6. 4). Identifying the variables as given in Eqs. (6. 8) through (6. 11) with Q = 0, we regain the Bonnor metric,[24]

$$ds^2 = \left(\frac{r^2 - a^2\cos^2\theta - 2Mr}{r^2 - a^2\cos^2\theta}\right)^2 dt^2$$
$$- \frac{(r^2 - 2Mr - a^2\cos^2\theta)^2 (r^2 - a^2\cos^2\theta)^2}{(r^2 - 2Mr - a^2\cos^2\theta + M^2\sin^2\theta)^3}\left(\frac{dr^2}{r^2 - 2Mr - a^2} + d\theta^2\right)$$
$$- \frac{(r^2 - a^2\cos^2\theta)^2 (r^2 - 2Mr - a^2)}{(r^2 - 2Mr - a^2\cos^2\theta)^2}\sin^2\theta\, d\phi^2, \qquad (7.\ 15)$$

in the correct form. There is confusion in the literature[22,25,26] because of adherence to complex potentials rather than our "I" formalism, Eq. (7. 5). The electromagnetic potential is given by

$$\Psi = \frac{2Ma\cos\theta}{r^2 - a^2\cos^2\theta}, \qquad (7.\ 16)$$

up to a constant phase, revealing that the dipole moment is given by

$$\mathcal{M} = 2Ma. \qquad (7.\ 17)$$

Although Eq. (7. 15) does not possess nonsingular event horizons, and thereby is not a black hole solution, it can nevertheless be used for the exterior fields of strongly gravitating sources without spin.

We may further exploit the fundamental symmetry group (Sec. 3) in the following order, (1) $\xi \to e^{i\alpha}\xi$, $\zeta \to e^{i\beta}\zeta$, (2) $\xi \to \xi\cosh\chi + \zeta\sinh\chi$, $\zeta \to \xi\sinh\chi + \zeta\cosh\chi$, (3) $\xi \to e^{i(\gamma - \alpha)}\xi$, $\zeta \to \zeta$, resulting in

$$\xi \to e^{i\gamma}\xi\cosh\chi + e^{i(\beta+\gamma-\alpha)}\zeta\sinh\chi, \qquad (7.\ 18)$$

$$\zeta \to e^{i\alpha} \xi \sinh \chi + e^{i\beta} \zeta \cosh \chi . \qquad (7.\ 19)$$

The real constants α, β, χ, and γ are related to the phase of charge, phase of electromagnetic dipole moment, charge, and phase of mass. In all, there are six parameters: four physical (mass, electric charge, electric and magnetic dipole moments), one probably unphysical (magnetic monopole charge), and one unphysical (N in Eq. (3. 12)). Likewise, we may start with any member of the TS solutions, e. g. Eq. (6. 16), and find similar metrics, which are called the five-parameter solutions in the literature.[25, 26]

The gyromagnetic ratio of these solutions is the reciprocal of that of the Dirac electron,

$$\frac{\mathcal{M}}{J} = \frac{M}{Q} . \qquad (7.\ 20)$$

We mention that these solutions are essentially static. Angular momentum, if any, comes entirely from the rotation of the Poynting vector.[23] We have obtained approximate stationary extensions of Eqs. (7. 13) and (7. 14) by means of perturbation theory. Right now we are looking for their exact versions (see Figure).

We owe the basic ideas of this section to Tanabe[23].

In the case of axial symmetry and no electromagnetic field, it would be easier to begin with Eq. (5. 1). From the Lagrangian density[1]

$$\mathcal{L} = -\frac{1}{2}\rho f^{-2}\nabla f\cdot\nabla f + \frac{1}{2}\rho^{-1}f^{2}\nabla\omega\cdot\nabla\omega, \qquad (8.\ 1)$$

we obtain the field equations for f and ω,

$$f\nabla^{2}f = \nabla f\cdot\nabla f - \rho^{2}f^{4}\nabla\omega\cdot\nabla\omega, \qquad (8.\ 2)$$

$$\nabla\cdot(\rho^{-2}f^{2}\nabla\omega) = 0. \qquad (8.\ 3)$$

Defining[27]

$$E = \frac{\rho}{f} + I\omega, \qquad (8.\ 4)$$

(for the meaning of I, see Sec. 7), we obtain the following equation for E, which is exactly analogous to Eq. (2. 34) for $\mathcal{E}$ with $\Psi = 0$,[27]

$$\frac{1}{2}(E+\bar{E})\nabla^{2}E = \nabla E\cdot\nabla E. \qquad (8.\ 5)$$

Further, we define

$$E = \frac{1-X}{1+X}, \qquad (8.\ 6)$$

and obtain the field equation for X, Eq. (7. 10), again. As is mentioned in Sec. 7, solutions to Eq. (7. 10) may be directly obtained from known solutions of Eq. (4. 4), or else by using the symmetry, Eq. (7. 5). If X is of the form

$$X = a + Ib, \tag{8. 7}$$

then the metric functions are given by

$$f = \rho \frac{(a+1)^2 - b^2}{a^2 - b^2 - 1}, \tag{8. 8}$$

$$\omega = \frac{2b}{(a+1)^2 - b^2} \tag{8. 9}$$

Since Eq. (7. 10) is invariant under

$$X \to -X, \tag{8. 10}$$

f and ω may be transformed into

$$f \to f\left(\frac{\rho^2}{f^2} - \omega^2\right), \tag{8. 11}$$

$$\omega \to -\frac{\omega}{\frac{\rho^2}{f^2} - \omega^2}. \tag{8. 12}$$

On the other hand, transformation $X \to e^{I\chi}X$, where $e^{I\chi}$ is defined by Eq. (7. 6), yields

$$f \to f\left(1 + \frac{2a(\cosh\chi - 1) + 2b\sinh\chi}{(a+1)^2 - b^2}\right), \tag{8. 13}$$

$$\omega \to \frac{2(a\sinh\chi + b\cosh\chi)}{(a+1)^2 - b^2 + 2a(\cosh\chi - 1) + 2b\sinh\chi}. \tag{8. 14}$$

As an example, we perform this transformation on the Minkowski spacetime,

$$a = \frac{1+\rho}{1-\rho}, \quad b = 0. \tag{8. 15}$$

The result

$$f = 1 + \frac{1}{2}(1-\rho^2)(\cosh\chi - 1), \tag{8. 16}$$

$$\omega = \frac{(1-\rho^2)\sinh\chi}{2+(1-\rho^2)(\cosh\chi - 1)}, \tag{8. 17}$$

has line singularity on the z-axis.

We owe the basic idea of this section to Ref. 27. Similar method may be found in Ref. 28.

9 INTERCHANGE OF THE ROLLS OF t AND ϕ

In all of the foregoing argument, we have utilized the presence of a Killing vector, $\vec{\xi}_{(t)} = \partial/\partial t$, i. e., time translation invariance of the metric. In the case of axial symmetry, however, there exists another Killing vector, $\vec{\xi}_{(\phi)} = \partial/\partial\phi$. This fact can be exploited as follows.[29]

A metric of the form given by Eq. (5. 1) may also be written as

$$ds^2 = -\tilde{f}(d\phi + \tilde{\omega}dt)^2 + \tilde{f}^{-1}[e^{2\tilde{\gamma}}(dz^2 + d\rho^2) + \rho^2 dt^2] \tag{9. 1}$$

where

$$\tilde{f} = f\omega^2 - f^{-1}\rho^2, \tag{9. 2}$$

$$\tilde{\omega} = \tilde{f}^{-1} f\omega = \frac{\omega}{\omega^2 - f^{-2}\rho^2}, \tag{9. 3}$$

$$e^{2\tilde{\gamma}} = (\omega^2 - f^{-2}\rho^2)e^{2\gamma}. \tag{9. 4}$$

Complex electromagnetic potential is written as

$$\tilde{\Psi} = A_\phi + i\tilde{\Phi} \tag{9. 5}$$

the meaning of which should be obvious. Now, if all of these are independent of ϕ, i. e., if there is axial symmetry, then we can use all the techniques given in the foregoing sections. The Maxwell 2-form is given by

$$F = -(\tilde{f}^{-1}\rho\tilde{\Phi}_{,\rho} - A_{\phi,z}\tilde{\omega})\, dz \wedge dt$$

$$+ (\tilde{f}^{-1}\rho\tilde{\Phi}_{,z} + A_{\phi,\rho}\tilde{\omega})\, d\rho \wedge dt$$

$$- A_{\phi,t}\, d\phi \wedge dt + A_{\phi,\rho}\, d\rho \wedge d\phi$$

$$- A_{\phi,z}\, d\phi \wedge dz + \tilde{f}^{-1}\rho^{-1}e^{2\tilde{\gamma}}\tilde{\Phi}_{,t}\, dz \wedge d\rho. \quad (9.\ 6)$$

As an example, the Minkowski spacetime, $\tilde{\mathcal{E}} = \tilde{f} = -\rho^2$, is written as

$$\tilde{\xi} = \frac{1+\rho^2}{1-\rho^2}, \qquad \tilde{\zeta} = 0. \quad (9.\ 7)$$

The fundamental transformation, Eq. (3. 6), yields

$$ds^2 = -f dt^2 + f^{-1}(f^2(dz^2 + d\rho^2) + \rho^2 d\phi^2) \quad (9.\ 8)$$

and

$$(E_z{}^2 + B_z{}^2)^{1/2} = f^{-1}\sinh\chi \quad (9.\ 9)$$

where

$$f = \frac{1}{4}[\cosh\chi + 1 + \rho^2(\cosh\chi - 1)]^2 \quad (9.\ 10)$$

Near the z-axis this gives a constant electromagnetic field.[29] Known solutions plus a constant electromagnetic field may be obtained similarly.[29]

We mention in passing that the formalism given here may be used for time-dependent metrics, also. However, equations are to be solved in curved space. In the presence of Constraint No. 3, e. g., we must solve

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\tilde{\Psi}}{\partial\rho}\right) + \frac{\partial^2\tilde{\Psi}}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial}{\partial t}\left(e^{2\tilde{\gamma}}\frac{\partial\tilde{\Psi}}{\partial t}\right) = 0, \quad (9.\ 11)$$

where $\tilde{\gamma}$ depends on $\tilde{\Psi}$. Once a solution is known, we may readily use the symmetry group to effect other solutions.

10 CONCLUSION AND DISCUSSION

We have introduced two complex potentials ξ and ζ that uniquely determine a spacetime with stationary gravitational and electromagnetic fields. Mass and charge only manifest themselves as singularities of the field equations governing the potentials. In Newtonian limit, Re ζ and Im ζ are electric and magnetic potentials. While Re ξ reduces to usual Newtonian potential in this limit, Im ξ is of order r^{-2}, and signifies "inertial frame dragging" caused by the source's inner movement (spin, e. g.). The field equations for ξ and ζ are invariant under a group of transformations, which fact is utilized to automatically generate new solutions. Various cases are examined in which the equations are solvable, and are summarized in Figure.

Whereas exact solutions are important in astrophysics, the spacetime symmetries studied in this work may be of interest in elementary particle physics, also. In this respect, it may be of use to note that the potentials are so constructed that $\xi \to \xi^*$ and $\zeta \to \zeta^*$ under parity transformation $\varepsilon_{abc} \to -\varepsilon_{abc}$, as can be already seen in the definitions of $\mathcal{E}$ and Ψ, Eqs. (2. 14), (2. 20), (2. 21), (2. 32), and(2. 33). All the imaginary parts are with ε_{abc}. Charge conjugation is trivial, $\zeta \to -\zeta$. The equations are thus invariant under parity and charge conjugation separately. Time reversal means that $w_m \to -w_m$ in Eq. (2. 5), and hence $\vec{\tau} \to -\vec{\tau}$. Now, an important point is that only when $\zeta \to \zeta^*$ at the same time do the equations stay invariant under $\vec{\tau} \to -\vec{\tau}$. In short, time reversal should mean $\xi \to \xi^*$ <u>and</u> $\zeta \to \zeta^*$ for CPT invariant systems. This may suggest that, for elementary particles, there be some functional relationship between ξ and ζ, the simplest of such being

$$\zeta = c\xi, \tag{10. 1}$$

where c is a real constant. Conversely, if the system is such that Eq. (10. 1) holds, then absence of "N" in Eq. (3. 12) automatically yields absence of magnetic monopole charge. Furthermore, the Dirac gyromagnetic ratio, Eq. (5. 38), automatically results.

Whether this testifies to a profound bond between the Einstein and the Dirac theories, nobody knows.

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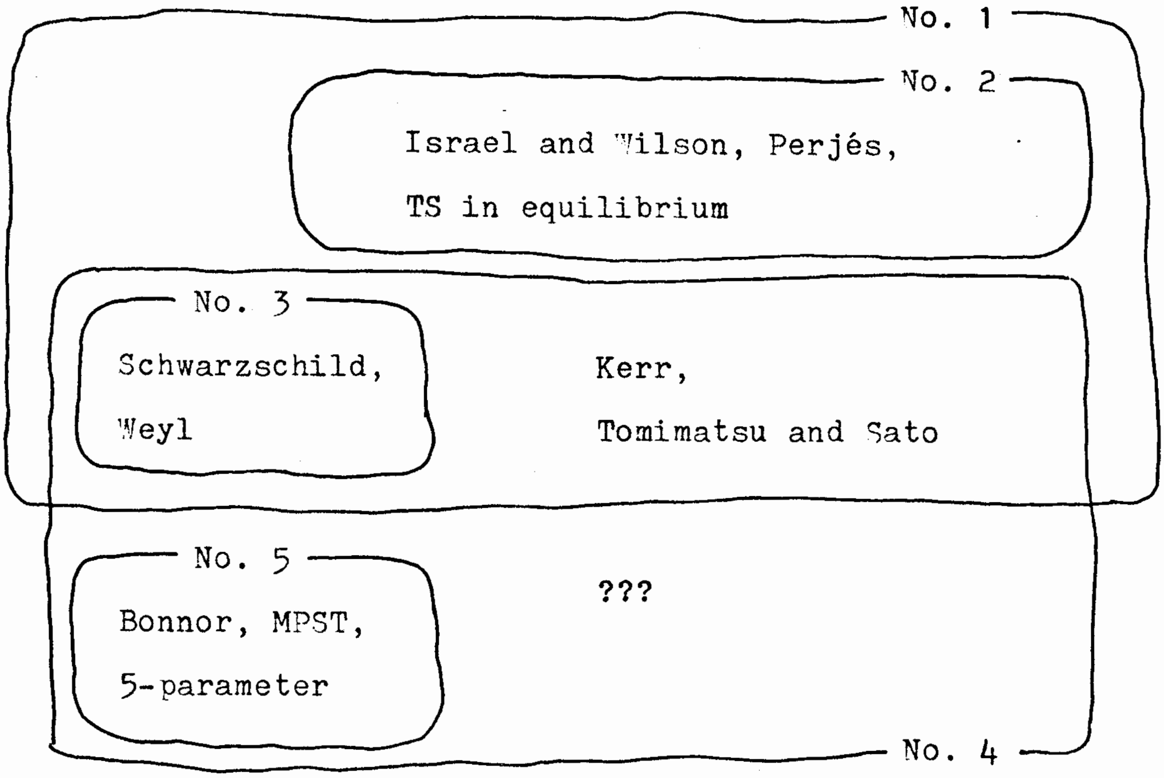
Figure



Five simple constraints, and known and obtained solutions.